A GENERAL METHOD OF SUMMATION OF DIVERGENT SERIES.

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INTRODUCTION.

In articles by Hardy and Chapman in the Quarterly Journal of Mathematics, vol. 42 (1911), and by Chapman in the Quarterly Journal, vol. 43 (1912), a general definition of summability of divergent series is given and its general properties discussed. Silverman also, in his dissertation "On the Definition of the Sum of a Divergent Series," discusses certain general definitions of summability. In the present paper, a similar general definition is given, but in such a form as to include as special cases most of the known particular methods, namely, the definitions of Cesàro, Hölder, Riesz, de la Vallée-Poussin, Plancherel, LeRoy, Borel's integral definition, Borel's exponential and generalized exponential methods, and the so-called Euler power series method. Certain general properties of this method are developed: the summability of convergent and properly divergent series, and the continuity, integration, and differentiation of uniformly summable series. Applications of these general results are then made to the various particular methods included as special cases under this general method.

THE GENERAL METHOD.

Let f_i be a function of the variables n and x, defined for all positive integral values of i, and for all positive values of n and x. Let $\sum\limits_0^\infty a_n$ be any given series, convergent or divergent.

If the expression $\sum\limits_{i=0}^{n} a_i f_i$ has a repeated double limit S:

$$\text{(I)} \qquad \underset{x}{\mathrm{L}}\, \underset{n}{\mathrm{L}} \sum_{i=0}^{n} a_i f_i(n, x) = S,$$

where $\underset{n}{\mathrm{L}}$ stands for $\lim\limits_{n\to\infty}$, we shall say that the series Σa_n is *summable* (A_f), by the summation-function $f_i(n, x)$, and that S is its *Sum* (or generalized sum).

In the above double limit, it is meant that n shall $\to \infty$ first, while x is held fixed, and then $x \to \infty$.

We shall say that a series of functions $\sum_{0}^{\infty} a_n(u)$ is *uniformly summable* (A_f) when the limit (I) is approached uniformly with respect to u in some interval (a, b); that is, when the simple limit $\mathbf{L}_n \sum_{i=0}^{n} a_i f_i(n, x) = S_x(u)$ say, is approached uniformly with respect to u in (a, b) and uniformly with respect to x for large values of x, and then the limit as $x \to \infty$: $\mathbf{L}_n S_x(u) = S(u)$, is approached uniformly with respect to u, according to the usual definition of uniform approach to a simple limit.

Theorem I. If two series Σa_n and Σb_n are summable (A_f) with Sums A and B, then the series $\Sigma(a_n \pm b_n)$ is summable (A_f) with the sum $A \pm B$.

The proof of this proposition is obvious.

Theorem II. If $\Sigma a_n(u)$ is uniformly summable (A_f) with respect to u in an interval (a, b), and if the terms $a_n(u)$ are continuous functions of u in (a, b), then its Sum $S(u)$ is a continuous function of u in (a, b).

This proposition is a direct consequence of the continuity of a uniform limit of continuous functions.

Theorem III. If $\sum_{0}^{\infty} a_n(u)$ is uniformly summable (A_f) in an interval (a, b) with Sum $S(u)$, and if the terms $a_n(u)$ are integrable in this interval, then the series obtained by integrating the given series term-by-term: $\sum_{0}^{\infty} \int_{c_1}^{c_2} a_n(u)du$, is summable (A_f), with Sum equal to $\int_{c_1}^{c_2} S(u)du$, where (c_1, c_2) is contained in (a, b).

If $\mathbf{L}_n^*$ means a uniform limit as $n \to \infty$, the theorem on integration of uniformly convergent series shows that the process $\mathbf{L}_n^*$ followed by the process $\int_{c_1}^{c_2}$ yields the same result as the process $\int_{c_1}^{c_2}$ followed by the process $\mathbf{L}_n^*$.

By hypothesis,

$$(a) \qquad S(u) = \mathbf{L}_x^* \left(\mathbf{L}_n^* \sum_{i=0}^{n} a_i(u) \cdot f_i(n, x) \right).$$

Then

$$(b) \qquad \int_{c_1}^{c_2} S(u)du = \int_{c_1}^{c_2} \mathbf{L}_x^* \left(\mathbf{L}_n^* \sum_{i=0}^{n} a_i(u) \cdot f_i(n, x) \right) du.$$

Remembering that $\int_{c_1}^{c_2}$ and $\sum_{i=0}^{n}$ are permutable processes, we obtain from (b),

$$(c) \qquad \int_{c_1}^{c_2} S(u)du = \mathop{\mathrm{L}}_{x}\left(\mathop{\mathrm{L}}_{n}\sum_{i=0}^{n} f_i(n, x)\cdot\int_{c_1}^{c_2} a_i(u)du\right),$$

which proves our theorem.*

Theorem IV. If $\Sigma a_n(u)$ is summable (A_f) with Sum $S(u)$, and if the terms $a_n(u)$ are differentiable, and if the series $\Sigma a_n'(u)$ obtained from $\Sigma a_n(u)$ by term-by-term differentiation, is uniformly summable (A_f) with respect to u in (a, b), with Sum $\sigma(u)$, and if the terms $a_n'(u)$ are integrable, then

$$\sigma(u) = \frac{d}{du} S(u).$$

This theorem may be proved by applying the theorem on integration to the series $\Sigma a_n'(u)$ in the same way as for the analogous case of uniformly convergent series.

So far, no restrictions whatever have been imposed upon the summation-function $f_i(n, x)$. In order to make summability (A_f) an actual generalization of convergency, we subject the function $f_i(n, x)$ to certain general restrictive conditions, and set up the following definition.

Definition. If $f_i(n, x)$ satisfies the following conditions:

$$(\mathrm{II}) \begin{cases} 1^\circ. \text{ When } n \text{ and } x \text{ are fixed, the sequence } (f_i) \text{ is positive and de-} \\ \qquad \text{creasing;} \\ 2^\circ. \mathop{\mathrm{L}}_{x}\mathop{\mathrm{L}}_{n} f_i(n, x) = 1 \text{ for } i \text{ fixed;} \end{cases}$$

and if limit (I) exists:

$$\mathop{\mathrm{L}}_{x}\mathop{\mathrm{L}}_{n}\sum_{0}^{n} a_i f_i(n, x) = S,$$

then we shall say that the series Σa_n is *summable* (A_f') with *Sum* S.

We then have the two following fundamental results:

Theorem V. If Σa_n is convergent with sum S, it is also summable (A_f') with Sum S.

For, let ϵ be any arbitrarily small positive number < 1, then since the given series is convergent, we can determine m so large that

$$(a) \qquad \left|\sum_{i=m}^{m+p} a_i\right| < \frac{\epsilon}{4}, \qquad (p = 1, 2, 3, \cdots);$$

let m now remain fixed, and then by 2°, integers X and N_X (depending on X) can be determined so large that

* This proof can be carried out using a weaker uniformity than that demanded in the definition given for uniform summability; for it is evidently sufficient to demand the uniformity of $\mathop{\mathrm{L}}_{n}\sum_{0}^{n} a_i(u)f_i(n, x)$ for every fixed x *separately* instead of requiring it for all x's simultaneously, and then, as before, the uniformity of $\mathop{\mathrm{L}}_{x}(\ )$.

(b) $\quad |f_i(n, x) - 1| < \frac{\epsilon}{4A}, \qquad \left\{ i = 0, 1, 2, \cdots, m-1;\ A \equiv \sum_0^{m-1} |a_i| \right\},$

for every $x > X$ and $n > N_X$, and

(c)
$$|f_m(n, x) - 1| < \frac{\epsilon}{4}$$

for every $x > X$ and $n > N_X$.

Write

$$\left| \sum_0^n a_i f_i(n, x) - S \right| \leqq \left| \sum_0^{m-1} a_i f_i - \sum_0^{m-1} a_i \right| + \left| \sum_m^\infty a_i \right| + \left| \sum_m^n a_i f_i \right|.$$

By (b) we have

$$\left| \sum_0^{m-1} a_i f_i - \sum_0^{m-1} a_i \right| < \frac{\epsilon}{4},$$

by (a),

$$\left| \sum_m^\infty a_i \right| < \frac{\epsilon}{4},$$

and by Abel's lemma, using 1° and (c),

$$\left| \sum_m^n a_i f_i \right| < |f_m(n, x)| \cdot \frac{\epsilon}{4} < \frac{\epsilon}{2}.$$

$$\therefore \left| \sum_0^n a_i f_i - S \right| < \epsilon$$

for every $x > X$ and $n > N_X$, hence it follows easily that

$$\underset{x}{\mathrm{L}}\ \underset{n}{\mathrm{L}} \sum_0^n a_i f_i = S.$$

Theorem VI. A properly divergent series is not summable (A_f') with finite Sum; that is, if $\underset{n}{\mathrm{L}} \sum_0^n a_i = +\infty$, then $\underset{x}{\mathrm{L}}\ \underset{n}{\mathrm{L}}\ a_i f_i = +\infty$.

If we put $\sum_{p=0}^{i} a_p \equiv s_i$, we have identically

(a)
$$\sum_{i=0}^n a_i f_i = \sum_{i=0}^{n-1} s_i (f_i - f_{i+1}) + s_n f_n.$$

Let K be any arbitrarily large positive number, then we may write

$$s_i = K + r_i \text{ for } i > m, \text{ where } r_i > 0.$$

Substituting this in (a), we get

$$\begin{aligned}\sum_{i=0}^{n} a_i f_i &= \sum_{i=0}^{m-1} (s_i - K)(f_i - f_{i+1}) + K \sum_{i=0}^{n-1} (f_i - f_{i+1}) \\ &\qquad + \sum_{i=m}^{n-1} r_i(f_i - f_{i+1}) + s_n f_n \qquad (b) \\ &= \sum_{i=0}^{m-1} (s_i - K)(f_i - f_{i+1}) + K \cdot f_0 + \sum_{i=m}^{n-1} r_i(f_i - f_{i+1}) + r_n f_n.\end{aligned}$$

Now keeping m fixed, and passing to the limit $\underset{x}{\mathrm{L}}\,\underset{n}{\mathrm{L}}$, applying the conditions 1°, 2°, and noting that the last two terms are always positive, we get

$$\underset{x}{\mathrm{L}}\,\underset{n}{\mathrm{L}} \sum_{0}^{n} a_i f_i > K,$$

and since K can be made as large as we please, it follows that

$$\underset{x}{\mathrm{L}}\,\underset{n}{\mathrm{L}} \sum_{0}^{n} a_i f_i = + \infty.$$

Particular Methods.

For the various particular methods included as special cases of our general definition of summability, we have the following summation-functions:

*Cesàro's Method:**

$$f_i(n, x) = \frac{n(n-1)\cdots(n-i+1)}{(k+n)\cdots(k+n-i+1)} \quad (i = 1, 2, 3, \cdots, n), \qquad f_0(n, x) = 1,$$

where k is any real number except a negative integer.

Hölder's Method:†

$$f_i(n, x) = \left(1 - \frac{i}{n+1}\right)^k \qquad (i = 0, 1, 2, \cdots, n),$$

where k is any real number.

Riesz's Method:‡

$$f_i(n, x) = \left\{1 - \frac{\lambda(i)}{\lambda(n)}\right\}^k (i = 1, 2, 3, \cdots, n), \qquad f_0(n, x) = 1,$$

where $\lambda(n)$ is a positive monotonic function of n, increasing to ∞ with n.

Vallée-Poussin's Method:||

$$f_i(n, x) = \frac{n(n-1)\cdots(n-i+1)}{(n+1)(n+2)\cdots(n+i)} \quad (i = 1, 2, \cdots, n), \qquad f_0(n, x) = 1.$$

* Cesàro, Bulletin des sciences math., sér. 2, vol. 14, pp. 114–120; Chapman, Proc. London Math. Soc., ser. 2, vol. 9, pp. 369–409; Knopp, Sitzungsberichte der Berliner Math. Ges., vol. 7, pp. 1–12.

† Hölder, Math. Annalen, vol. 20, pp. 535–549.

‡ Riesz, Paris Comptes Rendus, vol. 148, pp. 1658; vol. 149, pp. 18–21, 909–912.

|| Vallée-Poussin, Bull. de la Classe des Sciences de l'Académie Royale de Belgique (1908), pp. 193–254.

*Plancherel's Method:**

$$f_i(n, x) = \frac{n(n-1)\cdots(n-i+1)}{(n+2)(n+3)\cdots(n+i+1)} \; (i = 1, 2, \cdots, n), \qquad f_0(n, x) = 1.$$

Leroy's Method:†

$$f_i(n, x) = \frac{\Gamma(ie^{-1/x}+1)}{\Gamma(i+1)}.$$

Euler's Power Series Method:

$$f_i(n, x) = e^{-i/x}.$$

Borel's Integral Definition:‡

$$f_i(n, x) = \int_0^x e^{-t}\cdot\frac{t^i}{i!}\,dt.$$

Borel's Exponential Definition:§

$$f_i(n, x) = e^{-x}\{E_n(x) - E_{i-1}(x)\},$$

where

$$E_n(x) \equiv \sum_{i=0}^{n}\frac{x^i}{i!}.$$

Borel's Generalized Exponential Definition:||

$$f_i(n, x) = e^{-x^k}\{E_n(x^k) - E_{i-1}(x^k)\}.$$

It can be shown without much difficulty that all of these special summation functions, corresponding to the various particular methods discussed above, satisfy the conditions (II) imposed on the function $f_i(n, x)$ for summability (A_f'), and hence all of these special methods appear as special cases of our general definitions of summability (A_f) and (A_f'). If we now apply our general theorems I–VI to these various particular definitions of summability, we get a number of results, some new, and some already well-known, but the important point here is that they all follow at once from a few general propositions.

Thus, all convergent series are summable by all of the particular methods enumerated above, with a generalized Sum equal to the ordinary sum, and properly divergent series are not summable with finite Sum by any of these methods; and if a series of functions is uniformly summable by any of these methods, it has properties with respect to continuity, integration, and differentiation analogous to those for uniformly convergent series.

* Plancherel, Rend. Circ. Mat. di Palermo, vol. 33, pp. 41–66.

† LeRoy, Annales de Toulouse, sér. 2, vol. 2, pp. 317–430.

‡ Borel, Leçons sur les séries divergentes, p. 98.

§ Borel, Leçons sur les séries divergentes, p. 97.

|| Borel, Leçons sur les séries divergentes, p. 129.

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